Math 522 Exam 5 Solutions

1. Find the generating function for the sequence of squares $a_n = n^2$, i.e. $a_0 = 0^2, a_1 = 1^2, a_2 = 2^2, a_3 = 3^2, \ldots$. [Hint: $n^2 = 2\binom{n(n-1)}{2} + n$]

   Solution 1: We recall that $\frac{1}{(1-x)^3} = \sum_{n\geq 0}\binom{n+2}{2}x^n$, so $\frac{x^2}{(1-x)^3} = \sum_{n\geq 0}\binom{n}{2}x^n$, and that $\frac{x}{(1-x)^2} = \sum_{n\geq 0} nx^n$. Set $A(x) = \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} n^2x^n =$
   $= \sum_{n\geq 0}\left(2\binom{n(n-1)}{2} + n\right)x^n = 2\sum_{n\geq 0}\binom{n}{2}x^n + \sum_{n\geq 0} nx^n = 2\frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$.
   If desired, this can be simplified: $A(x) = \frac{2x^2+x(1-x)}{(1-x)^3} = \frac{x^2+x}{(1-x)^3}$.

   Solution 2: We begin with $\frac{x}{(1-x)^2} = \sum_{n\geq 0} nx^n$. Carefully aking derivatives of both sides, we get $\frac{1+x}{(1-x)^3} = \sum_{n\geq 0} n^2x^{n-1}$. Multiplying both sides by x we get $\frac{x+x^2}{(1-x)^3} = \sum_{n\geq 0} n^2x^n = \sum_{n\geq 0} a_nx^n$.

2. For all integers a, b, x, and p prime, prove that $1 \leftrightarrow (2a \vee 2b)$, i.e.:

   1. $(x-a)(x-b) \equiv 0 (\text{mod } p)$, if and only if

   2a. $x - a \equiv 0(\text{mod } p)$, or

   2b. $x - b \equiv 0(\text{mod } p)$.

   BONUS: Show by counterexample that the above need not hold if p isn't prime.

   $1 \leftarrow (2a \vee 2b)$: Proof by cases; we first prove $1 \leftarrow 2a$. Starting with $x - a \equiv 0(\text{mod } p)$, we use Thm. 4-2 and multiply by $(x-b) \equiv (x-b)$ to get $(x-a)(x-b) \equiv 0(x-b) = 0(\text{mod } p)$. $1 \leftarrow 2b$ is similar; combining we get $1 \leftarrow (2a \vee 2b)$.

   $1 \rightarrow (2a \vee 2b)$: We assume $(x-a)(x-b) \equiv 0 = 0(x-a)$. Now, either $(x-a) \equiv 0$ (and we're done) or $(x-a) \not\equiv 0$. But in this case $p \nmid (x-a)$ and so $\gcd(p, x-a) = 1$. We now use Thm. 4-3 to cancel $(x-a)$ from both sides, so $(x-b) \equiv 0$.

   BONUS: Take $p = 25$ (not prime), and $a = b = 0$. Then $x = 5$ satisfies $(x-a)(x-b) \equiv 0(\text{mod } p)$, but $x - a \not\equiv 0(\text{mod } p)$, and $x - b \not\equiv 0(\text{mod } p)$.

3. High score=103, Median score=75, Low score=50